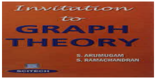
BEMA 56A Graph Theory



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UNIT I - Graphs and subgraphs

Sections :

1.1 Graphs

1.2 Degree of a vertex

1.3 Subgraphs

1.4 Isomorphism of graphs

1.5 Independent sets and Coverings.

Definition 1.

A graph $G = (V(G), E(G))$ consists of a non empty set,

$V(G)$ called the set of vertices of the graph,

$E(G)$ is the set of edges of the graph.

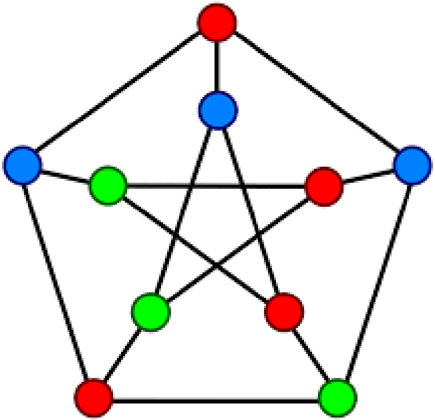


Figure : Petersen graph

Definition 2.

If $e = \{u, v\} \in E(G)$ then the edge e is said to join u and v.

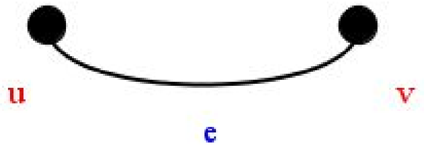


Figure : $e = uv$

Adjacent

Definition 3.

If $e = uv$ then the vertices u and v are adjacent.

(i.e., u and v join by an edge).

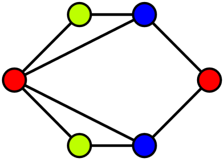


Figure : Adjacent vertices

Incident

Definition 4.

If $e = uv$ then the vertex u and the edge e are incident

with each other.

(p, q) - Graph

Definition 5.

A graph p vertices and q edges is called a (p, q)-graph.

Definition 6.

If two or more edges join same pair of vertices are called

multiple edges.

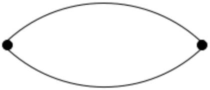


Figure : Parallel edges

Definition 7.

A graph allow multiple edges then the graph is called multi graph.

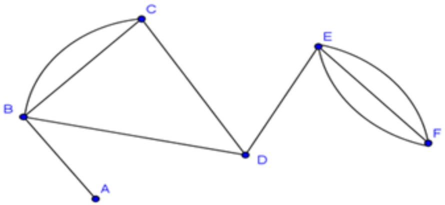
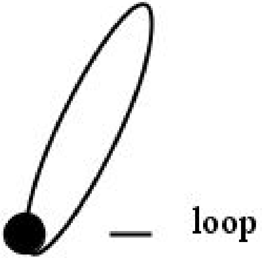


Figure : Multigraph

Definition 8.

Any edge joining a vertex to itself is called a loop.



Definition 9.

A graph allow multiple edge and loops then the graph

is called pseudo graph.

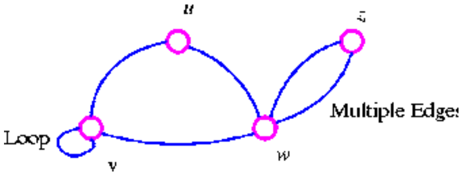


Figure : Pseudograph

Definition 10.

A graph contains no multiple edge and loops then

the graph is called simple graph.

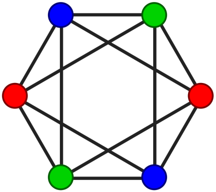


Figure : Simple graph

Null graph

Definition 11.

A graph whose edge set is empty is called a null graph

or totally disconnected graph.

Figure : Empty graph

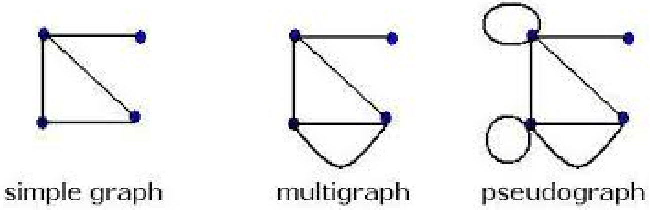


Figure : Graphs

Definition 12.

In a graph G any to vertices are adjacent is called a complete graph.

The complete graph with n vertices is denoted by K_n.

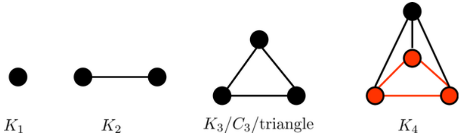


Figure : Complete graphs

Definition 13.

A graph G is called a bipartite graph

if vertex set V can be partitioned into two disjoint subsets V_1 and V_2

such that every edge of G joins a vertex of V_1 to a vertex of V_2.

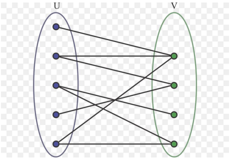


Figure : Bipartite graph

Definition 14.

In a bipartite graph G contains

every edge joining the vertices of V_1 to the vertices of V_2

then G is called a complete bipartite graph.

The complete bipartite graph with n vertices in V_1 and

m vertices in V_2 is denoted by $K_{n,m}$.

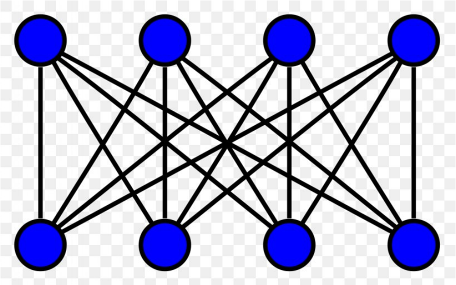


Figure : Complete bipartite graph

Definition 15.

The *degree* $d_G(v)$ of a vertex v in G is the number of edges of G incident with v, each loop counting as two edges.

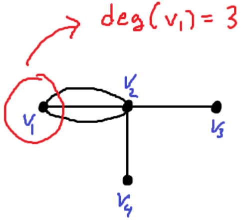


Figure : Degree at the vertex v_1

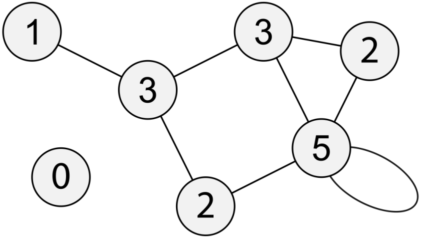


Figure : Compute degree at each vertex of the above graph

Definition 16.

A vertex v whose degree is zero is called isolated vertex.

Definition 17.

A vertex v whose degree is one is called end or pendent vertex.

Notation.

$\delta(G)$ denote the minimum degree of vertex in G.

$\Delta(G)$ denote the maximum degree of vertex in G.

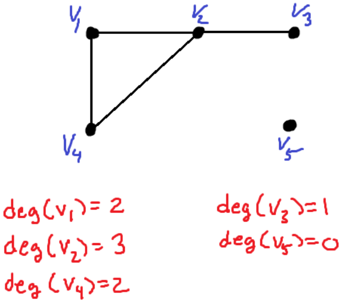


Figure : Isolated vertex and Pendent vertex

Definition 18.

A graph G is *k-regular* if $d_G(v) = k$ for all $v \in V(G)$.

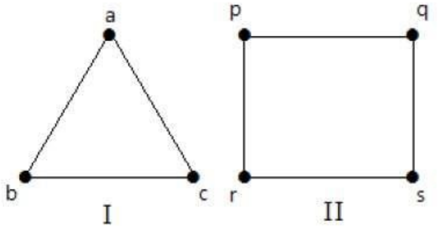


Figure : 2- regular graphs

Definition 19.

A graph G is *regular* if it is k-regular for some k.

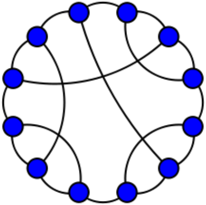


Figure : Regular graph

Remark :

1. The complete graph K_n is $(n-1)$-regular.
2. The complete bipartite graph $K_{n,n}$ is n-regular.
3. Here after graph means finite and simple.

Theorem 1.1.
The sum of the degrees of the vertices of a graph is equal to twice

the number of its edges.

(i.e., $\sum_{v \in V(G)} d_G(v) = 2q$).

Proof.

If $e = uv$ is an edge of G, e is counted once while counting the

degrees of each of u and v (even when $u = v$).

Hence, each edge contributes 2 to the sum of the degrees of the vertices.

Thus, the q edges of G contribute $2q$ to the degree sum.

Corollary 1.1.

In any graph G, the number of vertices of odd degree is even.

Proof.

By Theorem 1, $\sum_{v \in V(G)} d_G(v) = 2q$.

Let
$V_1 = \{v \in V(G) : d_G(v) \text{ is odd}\}$

$= \text{the set of vertices of odd degree in } G.$

$V_2 = \{v \in V(G) : d_G(v) \text{ is even}\}$

$= \text{the set of vertices of even degree in } G.$

Consider $\sum_{v \in V(G)} d_G(v) = \sum_{v \in V_1} d_G(v) + \sum_{v \in V_2} d_G(v)$.

By Euler theorem $2q = \sum_{v \in V_1} d_G(v) + \sum_{v \in V_2} d_G(v)$.

If $v \in V_2$, then $d_G(v)$ is even, $\implies \left(\sum_{v \in V_2} d_G(v) \right)$ is even.

$\implies$ an even number $= \sum_{v \in V_1} d_G(v) +$ an even number.

$\implies \left(\sum_{v \in V_1} d_G(v) \right)$ is even.

Since $v \in V_1 \Rightarrow d_G(v)$ is odd $\implies |V_1|$ is even.

i.e., the number of vertices of odd degree is even.

Definition 20.

A graph H is a **subgraph** of G, written $H \subseteq G$, if

$V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$ and

each edge of H has the same end vertices in H as in G.

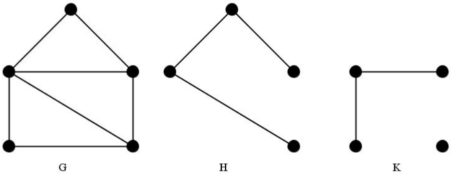


Figure : H and K are subgraphs of G

Proper subgraph, Supergraph

Definition 21.

When $H \subseteq G$ but $H \neq G$, write $H \subset G$ and call H a **proper subgraph** of G.

Definition 22.

If H is a subgraph of G, then G is a **supergraph** of H.

Definition 23.

A **spanning subgrph** of G is a subgraph H with $V(H) = V(G)$.

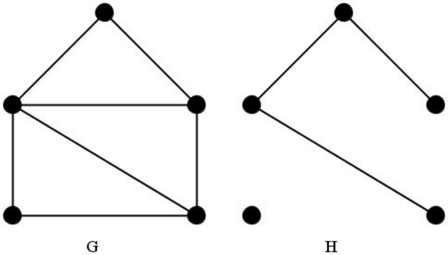


Figure : H- a spanning subgraph of G

Definition 24.

Suppose that V' is a nonempty subset of V.

The subgraph of G whose vertex set is V'.

Whose edge set is the set of those edges of G that have

both ends in V' is called the subgraph of **G induced by** $\mathbf{V'}$

and is denoted by $G[V']$.

We say that $G[V']$ is an induced subgraph of G.

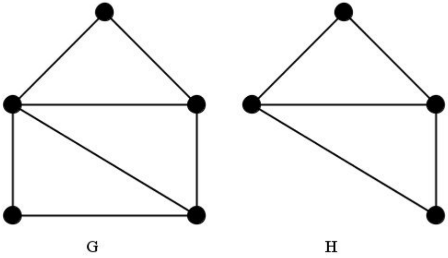


Figure : H - a vertex induced subgraph of G

The induced subgraph $G[V \setminus V']$ is denoted by $\mathbf{G} - \mathbf{V'}$.

It is the subgraph obtained from G by deleting the vertices in V'

together with their incident edges.

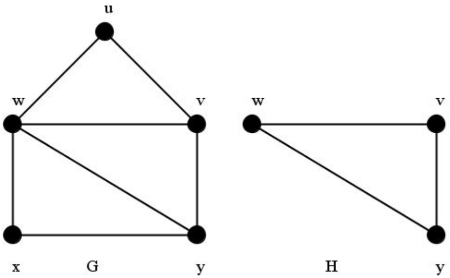


Figure : $H = G[V - \{u, x\}]$

In particular, if $V' = \{v\}$ we write $\mathbf{G} - \mathbf{v}$ for $\mathbf{G} - \{\mathbf{v}\}$.

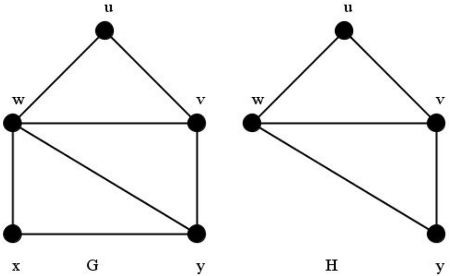


Figure : $H = G - \{x\}$

Edge induced subgraph

Definition 25.

Suppose that E' is a nonempty subset of E.

The subgraph of G whose vertex set is the set of ends of edges in E'.

Whose edge set is E' is called the subgraph of **G induced by E$'$**

and is denoted by $G[E']$.

We say that $G[E']$ is an edge-induced subgraph of G.

The induced subgraph of G with edge set $E \backslash E'$ is $\mathbf{G} - \mathbf{E}'$.

It is the subgraph obtained from G by deleting the edges in E'.

Similarly, the graph obtained from G by adding a set of edges E'

is denoted by $\mathbf{G} + \mathbf{E}'$.

If $E' = \{e\}$ we write $G - e$ and $G + e$ instead of

$\mathbf{G} - \{\mathbf{e}\}$ and $\mathbf{G} + \{\mathbf{e}\}$.

Definition 26. *Identical graphs*

Two graphs G and H are said to be *identical*, written $G = H$,

if $V(G) = V(H)$, $E(G) = E(H)$.

Definition 27.

Two graphs G and H are said to be **isomorphic,** written $G \cong H$, if there are bijections $\theta : V(G) \to V(H)$ and $\phi : E(G) \to E(H)$ such that $e = uv$ if and only if $\phi(e) = \theta(u)\theta(v)$.

Remark :

1. A bijection is a mapping which is both one-to-one and onto.
2. The pair (θ, ϕ) of mappings is called an isomorphism between G and H.

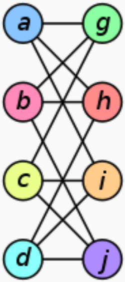
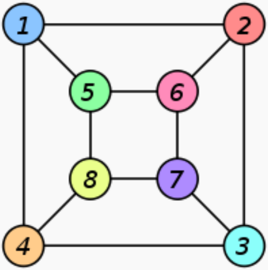
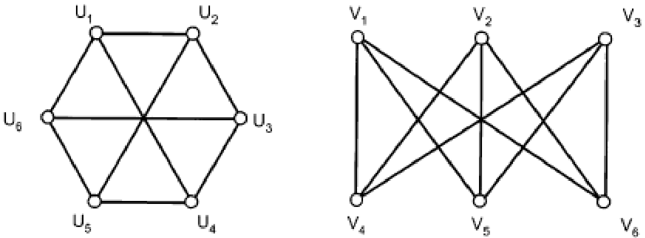
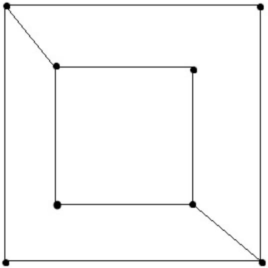
Graph G	Graph H	An isomorphism between G and H
		$f(a) = 1$ $f(b) = 6$ $f(c) = 8$ $f(d) = 3$ $f(g) = 5$ $f(h) = 2$ $f(i) = 4$ $f(j) = 7$

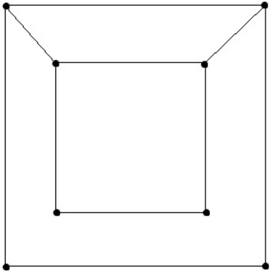
Figure : $G \cong H$

Prove that the pair of graphs below are isomorphic.

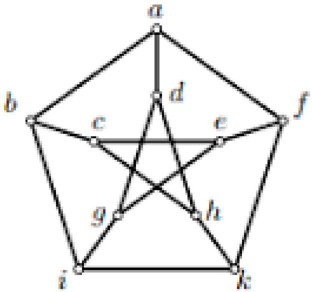


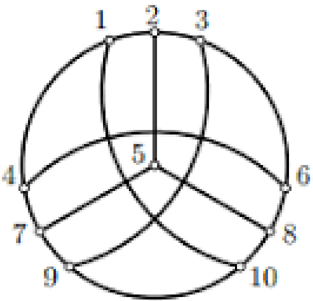
Verify whether the pair of graphs below are isomorphic or not?



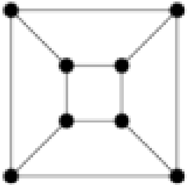


Verify whether the pair of graphs below are isomorphic or not?

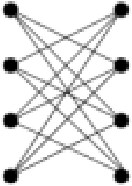




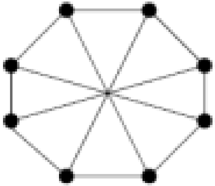
Verify whether the pairs of graphs below are isomorphic or not?



G_1



G_2



G_3

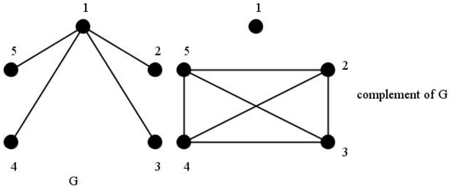
Complement of G

Definition 28. Let G be a given graph.

The **complement of** G is denoted by G^c or $\bar{G}$.

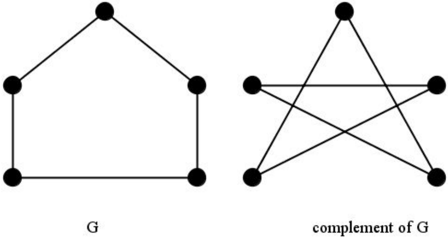
Two vertices are adjacent in G^c if and only if they are

non adjacent in G.



Definition 29.

A graph G is said to be a **self complementary** graph if $G \cong \bar{G}$.



Definition 30.

A **covering** or **vertex covering** of a graph G is a subset K of V such that every edge of G is incident with a vertex in K.

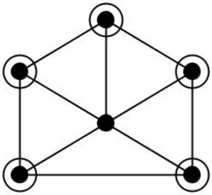


Figure : Whell graph W_5

Minimum covering

Definition 31.

A covering K is called a **minimum covering** if G has no covering K' with $|K'| < |K|$.

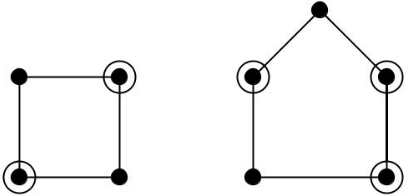


Figure : Cycles C_4 and C_5

Definition 32.

The number of vertices in a minimum covering of G is called

the **covering number** and it is denoted by β.

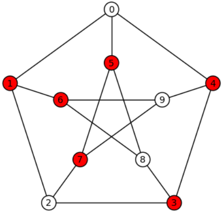


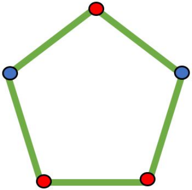
Figure : Petersen graph, $\beta = 6$

Independent set

Definition 33.

An **independent set** of a graph G is a subset S of V such that

no two vertices of S are adjacent in G.



Cycle graph of length 5

Definition 34.

An independent set S is called a **maximum** if G has no independent set S' with $| S' | > | S |$.

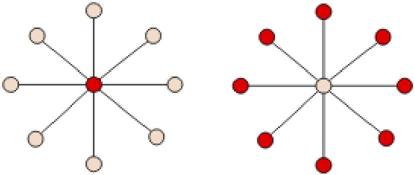


Figure : Independent set

Definition 35.

The number of vertices in a maximum independent set of G

is called the **independence number** and it is denoted by α.

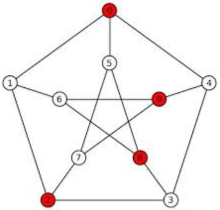


Figure : Petersen graph, $\alpha = 4$

Theorem 1.2.

A set $S \subseteq V$ is an independent set of G iff $V - S$ is a covering of G.

Proof.

By definition of independent set,

S is independent iff no two vertices of S are adjacent.

i.e., iff every edge of G is incident with at least one vertex of $V - S$.

i.e., iff $V - S$ is a covering of G.

Theorem 1.3.

$\alpha + \beta = p$.

Proof.

Let S be a maximum independent set of G.

Then $\mid S \mid = \alpha$.

Let K be a minimum covering of G.

Then $\mid K \mid = \beta$.

As S is independent, By theorem 1.2, $V - S$ is a covering of G.

K is a minimum covering of G, which implies $\mid K \mid \leq \mid V - S \mid$.

$\Rightarrow \beta \leq p - \alpha.$

$\Rightarrow \alpha + \beta \leq p. \qquad \longrightarrow \qquad (1)$

As K is covering, By theorem 1.2, $V - K$ is an independent set of G.

S is a maximum independent set of G, implies $\mid S \mid \geq \mid V - K \mid$.

$\Rightarrow \alpha \geq p - \beta.$

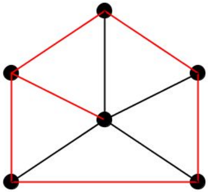
$\Rightarrow \alpha + \beta \geq p. \qquad \longrightarrow \qquad (2)$

From (1) and (2), we have

$\alpha + \beta = p.$

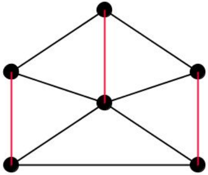
Definition 36.

An **edge covering** of a graph G is a subset F of E such that every vertex of G is incident with an edge in F.



Definition 37.

An edge covering $F^{'}$ is called a **minimum edge covering** if G has no edge covering $F^{''}$ with $\mid F^{''} \mid < \mid F^{'} \mid$.



Definition 38.

The number of edges in a minimum covering of G is called

the **edge covering number**. It is denoted by β'.

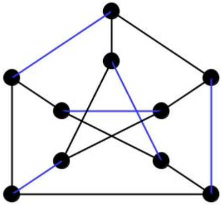
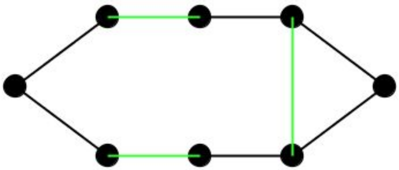


Figure : Petersen graph, $\beta' = 5$.

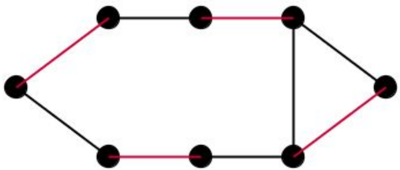
Definition 39.

A set of edges is called **edge independent** if no two of them are adjacent.



Definition 40.

An edge independent set S is called a **maximum** if G has no edge independent set S' with $|S'| > |S|$.



Definition 41.

The number of edges in a maximum edge independent set of G is called the **edge independence number**. It is denoted by α'.

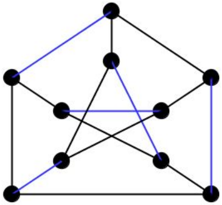


Figure : Petersen graph, $\alpha' = 5$.

Note .

It is not true that the complement of an independent set of edges

is a edge covering.

Theorem 1.4. (**Gallai**)

$\alpha' + \beta' = p$.

Proof.

Let S be a maximum independent set of edges of G.

Then $| S | = \alpha'$.

Let M be a set of edges, one incident edge for each of the $p - 2\alpha'$

vertices of G not covered by any edge of S.

Clearly, $(S \cup M)$, is a edge covering of G.

$\Rightarrow \mid S \cup M \mid \geq \beta'$.

$\Rightarrow \mid S \mid + \mid M \mid \geq \beta'$.

$\Rightarrow \alpha' + (p - 2\alpha') \geq \beta'$.

$\Rightarrow p - \alpha' \geq \beta'$.

$\Rightarrow p \geq \alpha' + \beta'. \qquad \longrightarrow (1)$

Now T be a minimum edge covering of G.

Then $\mid T \mid = \beta'$.

Claim: $G[T]$ the spanning subgraph of G induced by T,

is the union of stars.

Proof: Proof by contradiction.

Suppose $G[T]$ has an edge x such that both ends of x are also

incident with edges of T other than x.

Then $T - x$ will become a covering of G, a CONTRADICTION

to T is minimum.

Hence the claim is true.

Therefore, each edge of T is incident with at least one end vertex

of $G[T]$.

Let W be a set of vertices of $G[T]$ of exactly one end vertex for

each edge of T.

Hence $\mid W \mid = \mid T \mid = \beta'$.

Also, each vertex of $G[T]$ has exactly one vertex not in W.

Hence, $p = \mid W \mid +$ number of starts in $G[T]$.

i.e., $p = \beta' +$ number of starts in $G[T]$. $\longrightarrow (*)$

By choosing one edge from each star of $G[T]$,

we get a set Z independent edges of G.

But α' be the cardinality of a maximum independent set of edges

and Z is a set of independent edges.

$\Rightarrow \alpha' \geq \mid Z \mid$.

$\Rightarrow \alpha' \geq$ number of starts in $G[T]$.

By $(*)$, $p = \beta' +$ number of starts in $G[T]$.

$\Rightarrow p \leq \beta' + \alpha' \qquad \longrightarrow (2)$

From (1) and (2), we have $\alpha' + \beta' = p$.